17 Flux

A simple but very useful consequence of minimal surfaces being conformal harmonic immersions is the flux formula. Let M be a compact domain on a Riemann surface. According to Stoke's Theorem, for any C^2 function $f : M \to \mathbf{R}^n$,

$$\int_M \triangle_M f dA = \int_{\partial M} df(\vec{n})ds, \tag{17.66}$$

where dA is the element of area on M, $\triangle_M$ is the Laplacian on M, ds is the line element on ∂M, $\vec{n}$ is the outward unit normal vector to M along ∂M, and $df(\vec{n})$ is the directional derivative of f in the direction $\vec{n}$. Applying (17.66) to an isometric immersion $X : M \hookrightarrow \mathbf{R}^3$, we have that $dX(\vec{n})$ is the image in $\mathbf{R}^3$ of the outward *conormal* (i.e., $dX(\vec{n})$ is tangent to $X(M)$ but normal to $\partial X(M)$); writing $n^* = dX(\vec{n})$ we have

$$\int_M \triangle_M X \, dA = \int_{\partial M} n^* \, ds.$$

If X is minimal and M is equipped with the metric induced by X, then

$$\int_{\partial M} n^* \, ds = 0. \tag{17.67}$$

In particular, if $\vec{v}$ is any fixed vector in $\mathbf{R}^3$

$$\int_{\partial M} n^* \bullet \vec{v} \, ds = 0. \tag{17.68}$$

The integral in (17.68) can be thought of as the tangential part of the flux through $X(\partial M)$ of the flow in $\mathbf{R}^3$ with constant velocity vector $\vec{v}$. While (17.67) and (17.68) are quite simple and were undoubtedly known in the 19th century, they and their modifications have only recently come into widespread use in the study of minimal and constant mean curvature surfaces [43], [44].

As a sample application of the flux formula, we consider the catenoid. It was Euler who discovered the catenoid, the first nonplanar example of a minimal surface. He did this by finding the surface of revolution that was a critical point for the area functional. Consider a surface of revolution about the z-axis with profile curve $(r(t), t)$ in the xz-plane. Let S be the compact portion of the surface that is between $z = t_1$ and $z = t_2$. S is bounded by two circles of radii $r(t_1)$ and $r(t_2)$, respectively. The conormal of S at the level set $z = t$ is

$$\frac{1}{\sqrt{1 + r'(t)^2}}(r'(t)\cos\theta, \, r'(t)\sin\theta, \, 1).$$

Then computing the flux in the z-direction ($\vec{v} = (0, 0, 1)$), we get by (17.68)

$$\int_{S\cap\{z=t_1\}} \frac{1}{\sqrt{1 + r'(t_1)^2}} ds = \int_{S\cap\{z=t_2\}} \frac{1}{\sqrt{1 + r'(t_2)^2}} ds$$