Complex Variables: Single v Several

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0. PRELIMINARIES

Let D be a domain in the complex plane, i.e. an open and connected set in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is said to be *holomorphic* (or *analytic*) on D, if it is $\mathbb{C}$-differentiable at any point $z_0 \in D$, i.e. the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The n-dimensional complex space $\mathbb{C}^n$ is the product of n copies of the complex plain $\mathbb{C}$. It consists of n-tuples of the form $z = (z_1, \dots, z_n)$, $z_j \in \mathbb{C}$ for all j. We denote by $||z||$ the length of a vector z: $||z|| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

For a domain D lying in $\mathbb{C}^n$ we say that a function $f : D \to \mathbb{C}$ is holomorphic on D, if for every j and fixed $z_1, \dots, z_{j-1}, z_{j+1}, z_n$ it is holomorphic as a function of z_j (note the difference with real differentiability!).

A mapping $(f_1, \dots, f_n)$ between two domains in $\mathbb{C}^n$ is a holomorphic mapping, if each its component f_j is a holomorphic function.

Maximum Principle. If a function f is holomorphic on a domain $D \subset \mathbb{C}^n$ and continuous on $\overline{D}$, then

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

1. CONTINUATION PHENOMENON

The most impressive fact from complex analysis is the phenomenon of the continuation of functions (Hartogs, 1906; Poincaré, 1907). We elucidate its significance by an example. If a function $f(z_1, \dots, z_n)$ is defined and holomorphic in a neighbourhood of the boundary of a ball in the n-dimensional complex space $\mathbb{C}^n$, $n \geq 2$, then it turns out that $f(z_1, \dots, z_n)$ can be continued to a function holomorphic on the whole ball.