Asymptotic Solutions Of Scientific Interest

Roger J. Hosking
Department of Mathematics & Statistics
James Cook University of North Queensland
Townsville, Queensland, Australia. Qld 4811
Email: Roger.Hosking@jcu.edu.au

1 Introduction

The exact mathematical solution of many scientific problems is not feasible, and even when an exact solution can be found it may be difficult to interpret or inconvenient for numerical evaluation. An asymptotic solution may provide the insight we need however, and often enhances or complements any numerical computation that we might make. Thus the solution f sought may depend on some small parameter ϵ say, such that

$$f(\epsilon) \sim g_0(\epsilon), \qquad \epsilon \to 0$$

(equivalent to $\lim_{\epsilon\to 0} |f(\epsilon)/g_0(\epsilon)| = 1$) defines the solution behaviour by reference to a known function g_0. We say that "f is asymptotic to g_0" or "g_0 is an asymptotic approximation for f". Sometimes we can be more precise, and write f as an "asymptotic expansion"

$$f(\epsilon) \sim g_0(\epsilon) + g_1(\epsilon) + g_2(\epsilon) + \ldots, \qquad \epsilon \to 0$$

where $\{g_i(\epsilon)\}$ is a sequence of known functions such that $\lim_{\epsilon\to 0} |g_{j+1}(\epsilon)/g_j(\epsilon)| = 0 \ \forall j$. However, the function f usually depends on another scalar or vector variable x (say) independent of ϵ, so an asymptotic expansion may be invalid in certain regions – i.e. not *uniformly valid* (for all x). Such singular behaviour often arises when the domain of x is unbounded, or when the order or type of differential equation in f changes at the limit $\epsilon = 0$. There are various mathematical techniques to cope with singular behaviour, which can be an important feature to recognise and interpret in the mathematical sciences.

Another useful area of asymptotic analysis allows us to approximate analytically difficult integrals. Over 200 years ago Laplace evaluated the integral

$$F(\lambda) = \int_\alpha^\beta \exp[-\lambda f(t)] g(t) dt \qquad (\lambda > 0)$$

in the large parameter limit ($\lambda \to \infty$), when the known functions f and g are assumed sufficiently smooth. Thus if f has an absolute minimum at $t_0 \in (\alpha, \beta)$, we have

$$F(\lambda) \sim g(t_0) \exp[-\lambda f(t_0)] \sqrt{\frac{2\pi}{\lambda f''(t_0)}}, \qquad \lambda \to \infty. \tag{1}$$