Lecture 2: Stratifications in o-minimal structures

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Introduction

This note is devoted to the study of stratifications of definable subjects in o-minimal structures. The main results come from [**L1**]-[**L4**]. For the theory of stratifications, we refer the readers to [**Ma**],[**GPW**], [**T1**] and [**T2**].

In Section 1 we prove that the definable sets admit Verdier Stratification, and that the Verdier condition (w) implies the Whitney condition (b) in o-minimal structures. Note that the theorems were proved for subanalytic sets in [V] and [ŁSW] (see also [DW]), the former based on Hironaka's Desingularization, and the latter on Puiseux's Theorem. But, in general, these tools cannot be applied to sets belonging to o-minimal structures (e.g. to the set $\{(x, y) \in \mathbb{R}^2 : y = \exp(-1/x), x > 0\}$ in the structure generated by the exponential function).

In Section 2 we study the stratifications of definable functions. First we prove the existence of the stratifications of definable maps. Then we come to the existence of stratifications satisfying the Thom condition (a_f) for continuous functions definable in any o-minimal structures. In general, definable functions cannot be stratified to satisfy the strict Thom condition (w_f). However, if the structure is polynomially bounded, then its definable functions admit (w_f)-stratification. Our proof of this assertion is based on piecewise uniform asymptotics for definable functions from [M2], instead of Pawłucki's version of Puiseux's theorem with parameters, which is used in [KP] to prove the assertion for subanalytic functions.

Notations and Conventions. Throughout this note, let $\mathcal{D}$ denote some fixed, but arbitrary, o-minimal structure on $(\mathbb{R}, +, \cdot)$. "*Definable*" means definable in $\mathcal{D}$. Let p be a positive integer. If $\mathbb{R}^k \times \mathbb{R} \ni (y, t) \mapsto f(y, t) \in \mathbb{R}^m$ is a differentiable function, then $D_1 f$ denotes the derivative of f with respect to the first variables y. As usual, $d(\cdot, \cdot), \|\cdot\|$ denote the Euclidean distance and norm respectively. We will often use Cell decomposition theorem and Definable choice (see Lecture 1) in our arguments without citations. Submanifolds will always be embedded submanifolds.

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