CHAPTER 5

THE ALLARD REGULARITY THEOREM

Here we discuss Allard's ([AW1]) regularity theorem, which says roughly that if the generalized mean curvature of a rectifiable n-varifold $V = \underline{\underline{v}}(M,\theta)$ is in $L^p_{loc}(\mu_V)$ in U, $p > n$, if $\theta \geq 1$ μ_V a.e. in U, if $\xi \in \operatorname{spt} V \cap U$, and if $\omega_n^{-1} \rho^{-n} \mu_V(B_\rho(\xi))$ is sufficiently close to 1 for *some* sufficiently small* ρ, then V is *regular* near V in the sense that $\operatorname{spt} V$ is a $C^{1,1-n/p}$ n-dimensional submanifold near ξ.

A key idea of the proof is to show that V is well-approximated by the graph of a harmonic function near ξ. The background results needed for this are given in §20 (where it is shown that it is possible to approximate $\operatorname{spt} V$ by the graph of a Lipschitz function) and in §21 (which gives the relevant results about approximation by harmonic functions). The actual harmonic approximation is made as a key step in proving the central "tilt-excess decay" theorem in §22.

The idea of approximating by harmonic functions (in roughly the sense used here) goes back to De Giorgi [DG] who proved a special case of the above theorem (when $k = 1$ and when V corresponds to the reduced boundary of a set of least perimeter - see the previous discussion in §14 and the discussion in §37 below. Almgren used analogous approximations in his work [A1] for arbitrary $k \geq 1$. Reifenberg [R1, R2] used approximation by harmonic functions in a rather different way in his work or regularity of minimal surfaces.

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* Depending on $\|\underline{\underline{H}}\|_{L^p(\mu_V)}$