CHAPTER 2

SOME FURTHER PRELIMINARIES FROM ANALYSIS

Here we develop the necessary further analytical background material needed for later developments. In particular we prove some basic results about Lipschitz and BV functions, and we also present the basic facts concerning C^k submanifolds of Euclidean space. There is also a brief treatment of the area and co-area formula and a discussion of first and second variation formulae for C^2 submanifolds of Euclidean space. These latter topics will be discussed in a much more general context later.

§5. LIPSCHITZ FUNCTIONS

Recall that a function $f : X \to \mathbb{R}$ is said to be Lipschitz if there is $L < \infty$ such that (if d is the metric on X)

$$|f(x) - f(y)| \le L\, d(x,y) \qquad \forall\, x,y \in X .$$

$\mathrm{Lip}\, f$ denotes the least such constant L.

First we have the following trivial extension theorem.

5.1 THEOREM *If* $A \subset X$ *and* $f : A \to \mathbb{R}$ *is Lipschitz, then* $\exists\, \bar{f} : X \to \mathbb{R}$ *with* $\mathrm{Lip}\, \bar{f} = \mathrm{Lip}\, f$, *and* $f = \bar{f}|A$.

Proof Simply define

$$\bar{f}(x) = \inf_{y \in A}(f(y) + L\, d(x,y)) \ , \quad L = \mathrm{Lip}\, f \ .$$

Since $f(y) + L\, d(x,y) \ge f(z) - L\, d(x,z)$ $\forall\, x \in X$, $y,z \in A$, we see that $\bar{f}$