THE HILBERT TRANSFORM ON LIPSCHITZ CURVES[†]

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Our aim is to prove the following theorems.

THEOREM A. Let γ be a curve in the complex plane parametrized by $x + ih(x)$, $x \in \mathbb{R}$, where h is a real-valued absolutely continuous function with derivative $h' \in L_\infty(\mathbb{R})$. Let H_γ denote the Hilbert transform on γ , defined, for $u \in L_2(\gamma)$, by

$$(H_\gamma u)(x) = \frac{i}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{\sqrt{1+ih'(x)}\sqrt{1+ih'(y)}}{(x+ih(x)) - (y+ih(y))} u(y)dy \ .$$

Then $(H_\gamma u)(x)$ is defined for almost all $x \in \mathbb{R}$, and $H_\gamma u \in L_2(\mathbb{R})$. Indeed, there exists a constant c , depending only on $\|h'\|_\infty$, such that

$$\|H_\gamma u\|_2 \le c\|u\|_2 \qquad \text{for all } u \in L_2(\mathbb{R}) \ .$$

THEOREM B. Let $f \in L_\infty(\mathbb{R})$, with $\text{Re } f \ge 1$, and let M denote the maximal accretive operator in $L_2(\mathbb{R})$ defined by $Mu = -\frac{d}{dx}(f \frac{du}{dx})$, with domain $\mathcal{D}(M) = \{u \in H^1(\mathbb{R}) \mid f \frac{du}{dx} \in H^1(\mathbb{R})\}$ (where $H^1(\mathbb{R}) = \{u \in L_2(\mathbb{R}) \mid \frac{du}{dx} \in L_2(\mathbb{R})\}$). Then the domain of the square root, $M^{1/2}$, is $H^1(\mathbb{R})$, and, if $u \in H^1(\mathbb{R})$,

$$\tfrac{1}{6}\rho\|f\|_\infty^{-6}\|\tfrac{du}{dx}\|_2 \le \|M^{1/2}u\|_2 \le 6\rho\|f\|_\infty^{6}\|\tfrac{du}{dx}\|_2 \ .$$

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† This is an alternative treatment of results to be published in the Annals of Mathematics.