17. ALGORITHMS FOR FINITE RANK METHODS

As pointed out in Section 11, one approach to solving an eigenvalue problem for $T \in BL(X)$ is to consider a nearby operator $T_0 \in BL(X)$ which is simpler than T, and first solve an eigenvalue problem for T_0. For example, if T is a large full matrix then T_0 can be a much smaller matrix, or a matrix with some special structure like tridiagonality or sparsity. We can attempt to refine the eigenelements λ_0 and φ_0 of T_0 for obtaining approximations of corresponding eigenelements λ and φ of T. Several iterative procedures of this kind are given in Section 11 when λ_0 is a simple eigenvalue of T_0. In practice, one often chooses T_0 to be a bounded operator of finite rank. In the present section, we describe the step by step construction of the refinement schemes of Section 11 when $T_0 \in BL(X)$ is of finite rank. The relevant algorithms can be implemented on a computer. (Many of the results of this section appear in the thesis [DE].)

Let us first study the spectrum of a bounded operator T_0 of finite rank. Since T_0 is a compact operator, one can appeal to the well known results for the spectra of compact operators. However, we prefer to give an independent treatment.

If the dimension of X is greater than the rank of T_0 (in particular, if X is infinite dimensional), then $T_0 x = 0$ for some nonzero $x \in X$, i.e., 0 is an eigenvalue of T_0. Next, let $\lambda_0 \neq 0$. If λ_0 is not an eigenvalue of T_0, i.e., if $T_0 - \lambda_0 I$ is one to one, then we show that $T_0 - \lambda_0 I$ is also onto, so that λ_0 is not a spectral value of T_0. Let $\tilde{T} = (T_0 - \lambda_0 I)|_{R(T_0)}$. Since $R(T_0)$ is finite dimensional and $\tilde{T}$ is one to one, we see that $\tilde{T}$ maps $R(T_0)$