15. METHODS RELATED TO PROJECTIONS

In this and the next section we describe some concrete practical ways of constructing sequences of operators which approximate a compact operator T in the norm, or in the collectively compact manner. As such, they give resolvent operator approximations of T. The spectral considerations of the previous section are then applicable.

In the present section we consider a group of methods which arise from a sequence of (bounded) projections $\pi_n : X \to X$. For $T \in BL(X)$ and $n = 1,2,\ldots,$ we say that the operators

$$(15.1) \qquad T_n^P = \pi_n T \ , \quad T_n^S = T\pi_n \quad \text{and} \quad T_n^G = \pi_n T \pi_n$$

give the <u>projection method</u>, the <u>Sloan method</u> and the <u>Galerkin method</u> for approximating T, respectively. If each $\pi_n(X)$ is finite dimensional, then the above operators are of finite rank. We now consider the convergence of these approximation methods.

THEOREM 15.1 Let $\pi_n \xrightarrow{p} I$, the identity operator on X. Then (T_n^P), (T_n^S) and (T_n^G) are pointwise approximations of T.

If T is compact, then $T_n^P \xrightarrow{\| \ \|} T$, while $T_n^S \xrightarrow{cc} T$ and $T_n^G \xrightarrow{cc} T$.

If, in addition, $\pi_n^* \xrightarrow{p} I$, then $T_n^S \xrightarrow{\| \ \|} T$ and $T_n^G \xrightarrow{\| \ \|} T$. In particular, this is the case when X is a Hilbert space and each π_n is an orthogonal projection.

Proof It is easy to see that $T_n^P \xrightarrow{p} T$ and $T_n^S \xrightarrow{p} T$. Also, for $x \in X$,

$$\|T_n^G x - Tx\| \le \|\pi_n\| \ \|T_n^S x - Tx\| + \|T_n^P x - Tx\| \ .$$