14. RESOLVENT OPERATOR APPROXIMATION

As promised at the end of the last section, we introduce a mode of approximation which includes both the norm approximation and the collectively compact approximation, and which has nice implications for spectral approximation. At the end of this section we also show that the conditions needed for the convergence of various iteration schemes given in Section 11 are fulfilled under this mode of approximation. A variant of this mode has been called 'strong approximation' in the literature (cf. [CL], [LN]), but we have chosen to give it another more appropriate name.

A sequence (T_n) in $BL(X)$ is said to be a <u>resolvent operator approximation of</u> $T \in BL(X)$ if $T_n \xrightarrow{p} T$, and for every $z \in \rho(T)$,

$$\|(T-T_n)R(z)(T-T_n)\| \to 0 \ . \tag{14.1}$$

We denote this fact by $T_n \xrightarrow{ro} T$. Showing that a sequence (T_n) is a resolvent operator approximation of T is, in general, a formidable task. However, there are two well known modes of approximation which imply the resolvent operator approximation. It is obvious that $T_n \xrightarrow{\|\ \|} T$ implies $T_n \xrightarrow{ro} T$. Also, it follows by letting $A = TR(z)$, $A_n = T_nR(z)$ for $z \in \rho(T)$, and $B = T$, $B_n = T_n$ in Proposition 13.3 that $T_n \xrightarrow{cc} T$ implies $T_n \xrightarrow{ro} T$.

Let $T_n \xrightarrow{p} T$. By the uniform boundedness principle ([L], 9.1 and 9.3),

$$\|T\| \leq \sup\{\|T_n\| : n = 1,2,\ldots\} < \infty \ .$$

For a closed subset E of $\rho(T)$, we have

$$\max_{z\in E} \|R(z)\| < \infty \ , \quad \upsilon_n(E) \equiv \max_{z\in E} \|(T-T_n)R(z)\| < \infty \ ,$$

$$\upsilon(E) \equiv \sup_{n=1,2,\ldots} \upsilon_n(E) < \infty \ , \tag{14.2}$$