13. APPROXIMATION OF BOUNDED OPERATORS

In this section we consider some modes of approximating an operator $T \in BL(X)$ by a sequence (T_n) of operators in $BL(X)$. We study the relationships among these modes. Our interest lies in the approximation of $\sigma(T)$ by $\sigma(T_n)$.

If $T_n x \to Tx$, $x \in X$, i.e., $\|T_n x - Tx\| \to 0$ for every $x \in X$, we say that (T_n) <u>is a pointwise approximation of</u> T, and denote this fact by $T_n \xrightarrow{p} T$.

The pointwise approximation has, in general, no implication for the approximation of the spectrum: (i) For $n = 1,2,\ldots$, there may exist an eigenvalue λ_n of T_n such that (λ_n) converges to an element of the resolvent set of T. For example, let $X = \ell^2$, and for $x = [x(1), x(2),\ldots]^t$ in X, let

$$T_n x = [0,\ldots,0,\ x(n+1),\ x(n+2),\ldots]^t ,$$

where the zeros occur in the first n places. Then $T_n \xrightarrow{p} T = 0$, $\sigma(T_n) = \{0,1\}$, while $\sigma(T) = \{0\}$. (ii) There may exist an eigenvalue λ of T such that no subsequence of (λ_n), where $\lambda_n \in \sigma(T_n)$, converges to λ. For example, let $X = \ell^2$, and

$$T_n x = [x(2),\ldots,\ x(n),\ 0,\ 0,\ \ldots]^t , \quad Tx = [x(2),\ x(3),\ \ldots]^t .$$

Then $T_n \xrightarrow{p} T$, $\sigma(T_n) = \{0\}$, while $\sigma(T) = \{z \in \mathbb{C} : |z| \le 1\}$ and every $z \in \mathbb{C}$ with $|z| < 1$ is an eigenvalue of T.

The above two examples point out the lack of upper semicontinuity and lower semicontinuity of the spectrum with respect to the pointwise approximation. (Cf. (9.11) and the discussion there.) However, for self-adjoint operators on a Hilbert space, we do have lower semicontinuity of the spectrum with respect to the pointwise convergence.