12. FINITE DIMENSIONAL EIGENVALUE PROBLEM

This section is devoted to a review of some important methods of finding eigenelements of an operator $T : \mathbb{C}^n \to \mathbb{C}^n$. Let T be represented by the $n \times n$ matrix $[t_{i,j}]$ with respect to the standard basis $e_1,\ldots,e_n$ of $\mathbb{C}^n$. We shall denote this matrix also by the letter T . Then $T^* = [\bar{t}_{j,i}] = T^H$.

Decomposition Results

Before we discuss the matrix eigenvalue problem, we describe some decompositions of a matrix. The motivation for these results comes from the following facts. If T is a <u>diagonal</u> <u>matrix</u> (i.e., $t_{i,j} = 0$ if $i \neq j$) , then clearly the diagonal entries are the eigenvalues of T with $e_1,\ldots,e_n$ as the corresponding eigenvectors. Next, if T is an <u>upper</u> <u>triangular</u> <u>matrix</u> (i.e., $t_{i,j} = 0$ if $i > j$), then again the diagonal entries are the eigenvalues of T , but for a fixed i , e_i is not an eigenvector (corresponding to $t_{i,i}$) unless $t_{i,j} = 0$ for all $j > i$. If T is partitioned as

$$T = \begin{array}{cc} \left[\begin{array}{cc} T_{1,1} & T_{1,2} \\ & \\ 0 & T_{2,2} \end{array}\right] & \begin{array}{l} k \\ \\ n-k \end{array} \\ \begin{array}{cc} k & \quad n-k \end{array} & \end{array} , \tag{12.1}$$

then the eigenvalues of T consist of the eigenvalues of $T_{1,1}$ and of $T_{2,2}$, since $\det(T-zI_n) = \det(T_{1,1}-zI_k)\det(T_{2,2}-zI_{n-k})$.

Also, if U is a <u>unitary</u> <u>matrix</u>, (i.e., $U^HU = I = UU^H$) , then the eigenvalues of T and of U^HTU are the same; if x is an eigenvector of U^HTU corresponding to λ , then Ux is a corresponding eigenvector of T .