10. RAYLEIGH-SCHRÖDINGER SERIES

Let λ_0 be a simple eigenvalue of $T_0 \in BL(X)$ and φ_0 be a corresponding eigenvector. For $V_0 \in BL(X)$, consider the family of operators $T(t) = T_0 + tV_0$, $t \in \mathbb{C}$. For suitable values of t, we develop an iterative procedure of obtaining an eigenvalue $\lambda(t)$ of $T(t)$, and a corresponding eigenvector $\varphi(t)$ starting with the initial terms λ_0 and φ_0. We give conditions on t for which this procedure is guaranteed to converge. We also discuss the question of the simplicity of $\lambda(t)$, and of its isolation from the rest of $\sigma(T(t))$. The theory of linear perturbation developed in the last section will be heavily relied on.

Since λ_0 is a *simple* eigenvalue of T_0 with a corresponding eigenvector φ_0, it follows from Theorem 8.3 that there is an eigenvector φ_0^* of T_0^* corresponding to the eigenvalue $\bar{\lambda}_0$ such that $\langle\varphi_0,\varphi_0^*\rangle = 1$, and that the spectral projection P_0 associated with T_0 and λ_0 is given by

(10.1) $$P_0x = \langle x,\varphi_0^*\rangle\varphi_0 \ , \ x \in X \ .$$

The reduced resolvent S_0 associated with T_0 and λ_0 satisfies

(10.2) $$S_0 = \lim_{z\to\lambda_0} R_0(z)(I-P_0) \ .$$

Let Γ be a curve in $\rho(T_0)$ which isolates λ_0 from the rest of $\sigma(T_0)$. Then Corollary 9.9 shows that for all t in the disk

(10.3) $$\partial_\Gamma = \{t \in \mathbb{C} : |t| < 1/\max_{z\in\Gamma} r_\sigma(V_0R_0(z))\} \ ,$$

the operator $T(t)$ has only one spectral value $\lambda(t)$ inside Γ, it is a simple eigenvalue of $T(t)$, and $t \mapsto \lambda(t)$ is an analytic