9. LINEAR PERTURBATION

In this section we study the effect on the spectrum of an operator $T_0 \in BL(X)$ when it is subjected to a perturbation $V_0 \in BL(X)$. Thus, if we denote the perturbed operator $T_0 + V_0$ by T, we wish to obtain information about $\sigma(T)$ when $\sigma(T_0)$ is known. In this process we shall attempt to allow as 'large' a perturbation V_0 as possible.

We start our investigation by considering the invertibility of an operator which is close to an invertible operator.

We first note that if A and B are both invertible operators in $BL(X)$, then

$$(9.1) \qquad B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1} = A^{-1}(A-B)B^{-1} .$$

More generally, if $z \in \rho(A) \cap \rho(B)$, then

$$(9.2) \qquad \begin{aligned} R(B,z) - R(A,z) &= R(B,z)(A-B)R(A,z) \\ &= R(A,z)(A-B)R(B,z) . \end{aligned}$$

This follows on replacing A by $A - zI$ and B by $B - zI$ in (9.1). The relation (9.2) is known as the <u>second resolvent identity</u>.

THEOREM 9.1 Let $A, B \in BL(X)$ and A be invertible. Let

$$(9.3) \qquad r_\sigma((A-B)A^{-1}) < 1 .$$

Then B is invertible, and

$$(9.4) \qquad B^{-1} = A^{-1} \sum_{k=0}^{\infty} [(A-B)A^{-1}]^k = \sum_{k=0}^{\infty} [A^{-1}(A-B)]^k A^{-1} .$$

If , in fact,

$$(9.5) \qquad \|[(A-B)A^{-1}]^2\| < 1 ,$$