7. ISOLATED SINGULARITIES OF R(z)

In the last section we have considered the Laurent expansion of the resolvent operator $R(z)$ in an annulus contained in the resolvent set $\rho(T)$ of $T \in BL(X)$. We now specialize to the case when the inner circle of such an annulus degenerates to a point λ; i.e., when a punched disk $\{z \in \mathbb{C} : 0 < |z-\lambda| < \delta\}$ lies in $\rho(T)$. Let Γ be any curve in $\rho(T)$ such that $\sigma(T) \cap \operatorname{Int} \Gamma \subset \{\lambda\}$. Since the operators $P_\Gamma(T)$, $S_\Gamma(T,\lambda)$ and $D_\Gamma(T,\lambda)$ do not depend on Γ, we denote them simply by P_λ, S_λ and D_λ, respectively. The operators S_λ and D_λ have special features. By the first resolvent identity (5.5), we have

$$\begin{aligned} S_\lambda &= \frac{1}{2\pi i}\int_\Gamma \frac{R(w)}{w-\lambda}\,dw \\ &= \lim_{z\to\lambda} \frac{1}{2\pi i}\int_\Gamma \frac{R(w)}{w-z}\,dw \\ &= \lim_{z\to\lambda} \frac{1}{2\pi i}\int_\Gamma \frac{R(z)+R(w)-R(z)}{w-z}\,dw \\ &= \lim_{z\to\lambda} \frac{1}{2\pi i}\left[R(z)\int_\Gamma \frac{dw}{w-z} + \int_\Gamma \frac{(w-z)R(z)R(w)}{w-z}\,dw\right] \\ &= \lim_{z\to\lambda}\left[R(z) + R(z)(-P)\right]. \end{aligned}$$

Thus, we see that

$$(7.1)\qquad S_\lambda = \lim_{z\to\lambda} R(z)(I-P).$$

Next, it follows by Proposition 6.4 and (5.1) that

$$(7.2)\qquad \sigma(S_\lambda) \subset \{0\} \cup \{1/(\mu-\lambda) : \mu \in \sigma(T),\ \mu \neq \lambda\},$$

where the inclusion is proper if and only if $\lambda \notin \sigma(T)$. Hence

$$(7.3)\qquad r_\sigma(S_\lambda) = \frac{1}{\operatorname{dist}(\lambda, \sigma(T)\setminus\{\lambda\})}.$$

Again, Proposition 6.4 implies that

$$(7.4)\qquad \sigma(D_\lambda) = \{0\} \text{ and } r_\sigma(D_\lambda) = 0.$$