6. SPECTRAL DECOMPOSITION

In this section we develope a powerful method of decomposing an operator $T \in BL(X)$ in such a way that the spectrum $\sigma(T)$ of T becomes the *disjoint* union of the spectra of the restrictions of T. It also allows us to determine the coefficients in the Laurent expansion of the resolvent operator $R(z)$. We start with a simple result.

PROPOSITION 6.1 Let $T \in BL(X)$ be decomposed by (Y,Z). Then

$$\rho(T) = \rho(T_Y) \cap \rho(T_Z) , \tag{6.1}$$

or, equivalently

$$\sigma(T) = \sigma(T_Y) \cup \sigma(T_Z) . \tag{6.2}$$

In fact, for z in $\rho(T)$, we have

$$R(T,z)|_Y = R(T_Y,z) \quad \text{and} \quad R(T,z)|_Z = R(T_Z,z) , \tag{6.3}$$

while for $z \in \rho(T_Y) \cap \rho(T_Z)$, we have

$$R(T_Y,z)P + R(T_Z,z)(I-P) = R(T,z) , \tag{6.4}$$

where P is the projection on Y along Z.

Proof The formula (6.3) can be verified easily and since P commutes with T (Proposition 2.1) the formula (6.4) also follows. Hence the relations (6.1) and (6.2) hold. //

We remark that when $T = T_Y \oplus T_Z$, $\sigma(T)$ need not be the *disjoint* union of $\sigma(T_Y)$ and $\sigma(T_Z)$, since the parts T_Y and T_Z of T can, in general, have common spectral values. The simplest example is given