5. RESOLVENT OPERATORS

In this section we define the spectrum and the resolvent set of $T \in BL(X)$. The analyticity and the power series expansion of the resolvent operator are the main considerations. We also obtain the spectral radius formula. This section lays the basis of the spectral theory.

Let $T \in BL(X)$. The resolvent set of T is defined and denoted as follows:

$$\rho(T) = \{z \in \mathbb{C} : T - zI \text{ is invertible in } BL(X)\} .$$

The spectrum of T is the complement of $\rho(T)$ in $\mathbb{C}$, and is denoted by $\sigma(T)$. It follows by the *open mapping theorem* ([L], 11.1) that $\lambda \in \sigma(T)$ if and only if either $T - \lambda I$ is not one to one, or it is not onto X. For $z \in \rho(T)$, the operator

$$R(T,z) = (T-zI)^{-1}$$

is called the resolvent operator of T at z. When there is no confusion possible, we shall denote it simply by $R(z)$.

It can be observed immediately that for any $z_0 \in \mathbb{C}$,

$$\sigma(T+z_0I) = \{\lambda + z_0 : \lambda \in \sigma(T)\} ,$$

and if $0 \notin \sigma(T)$, i.e., if T is invertible, then

$$\sigma(T^{-1}) = \{1/\lambda : \lambda \in \sigma(T)\} .$$

It then follows that for $z_0 \in \rho(T)$,

$$\sigma(R(z_0)) = \{1/(\lambda-z_0) : \lambda \in \sigma(T)\} . \tag{5.1}$$

In fact, it can be readily verified that for $z \in \rho(T)$ and $z \neq z_0$,