4. BANACH SPACE-VALUED ANALYTIC FUNCTIONS

In this section we generalize the theory of complex-valued analytic functions of a complex variable by considering functions with values in a complex Banach space Y . The reason for considering the letter Y instead of the usual letter X is that we shall later consider $Y = BL(X)$, where X is a given complex Banach space.

Throughout this section D will denote a nonempty open connected set in $\mathbb{C}$.

A function $f : D \to Y$ is said to be <u>analytic on</u> D if for every $z_0 \in D$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in Y ; it then will be denoted by $f'(z_0)$ and called the <u>derivative of</u> f <u>at</u> z_0 .

If f is analytic on D and if $y^* \in Y^*$, then it follows from the conjugate linearity and the continuity of y^* that the map $z \mapsto \langle f(z), y^* \rangle$ is a complex-valued analytic function for z in D , and

$$\langle f(\cdot), y^* \rangle'(z) = \langle f'(z), y^* \rangle . \tag{4.1}$$

Dunford's theorem states the amazing fact that if $z \mapsto \langle f(z), y^* \rangle$ is analytic for z in D for every $y^* \in Y^*$, then, in fact, f is analytic on D ([L], 9.5). This result will allow us to transfer many interesting formulae from the theory of $\mathbb{C}$-valued functions to the case of Y-valued functions.

Before we discuss the integration of Y-valued functions, we deduce some useful results for Y-valued analytic functions.