3. FINITE DIMENSIONALITY

In any *numerical approximation* process, we deal solely with finite dimensional subspaces and with operators whose ranges are finite dimensional. In this section we study such subspaces and operators.

We start with a result concerning the closedness of the sum of two closed subspaces of a complex Banach space X. In general, such a sum need not be a closed subspace, as can be seen by considering $X = \ell^2$, F_1 = the closed linear span of $\{e_{2n} : n = 1,2,\dots\}$, i.e., $\left\{\sum_{n=1}^{\infty} a_n e_{2n} : a_n \in \mathbb{C}\ ,\ \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\}$ and F_2 = the closed linear span of $\{e_{2n} + \frac{1}{n} e_{2n+1} : n = 1,2,\dots\}$, i.e., $\left\{\sum_{n=1}^{\infty} b_n(e_{2n} + \frac{1}{n} e_{2n+1}) : b_n \in \mathbb{C}\ ,\ \sum_{n=1}^{\infty} |b_n|^2 < \infty\right\}$. Then $\sum_{n=1}^{j} \frac{e_{2n+1}}{n}$ belongs to $F_1 + F_2$ for each $j = 1,2,\dots,$ but $\sum_{n=1}^{\infty} \frac{e_{2n+1}}{n}$ does not. However, if one of the summands is finite dimensional, we have the following result.

PROPOSITION 3.1 Let Y be a finite dimensional subspace and Z be a closed subspace of X. Then $Y + Z = \{y + z : y \in Y,\ z \in Z\}$ is a closed subspace of X. In particular, Y itself is closed in X.

Proof Assume first that Y is one dimensional, say $Y = \text{span}\{y_1\}$. If $y_1 \in Z$, then $Y + Z = Z$, which is given to be closed. If $y_1 \notin Z$, let

$$d = \text{dist}(y_1, Z) > 0 .$$

Consider a sequence $(\alpha_n y_1 + z_n)$ in $Y + Z$, which converges to x in X. Now, for every $z \in Z$, we have

$$|\alpha_n| d \leq \|\alpha_n y_1 + z\| . \tag{3.1}$$