2. PROJECTION OPERATORS

A projection operator allows us to decompose a Banach space X as well as a commuting bounded operator T on X. In this way, we are able to concentrate only on a 'part' of X, or of T. These projection operators will often occur in the spectral theory as well as in various approximation procedures that we shall study.

A complex Banach space X is said to be decomposed by a pair (Y,Z) of its closed subspaces if $X = Y + Z$ and $Y \cap Z = \{0\}$. In this case, we write

$$X = Y \oplus Z .$$

This happens if and only if every $x \in X$ can be written in a unique way as $y + z$ with $Y \in Y$ and $z \in Z$; if we let $Px = y$, then P is a linear map from X to X and satisfies $P^2 = P$, i.e., P is a projection. Also, the set $\{(x,Px) : x \in X\}$ is closed in $X \times X$. This can be seen as follows. Let $x_n \to x$ and $Px_n \to y$. Since $Px_n \in Y$ and Y is closed, we see that $y \in Y$. Also, $x_n - Px_n \in Z$ and Z is closed, so that $x - y \in Z$. Since $x = y + (x-y)$ with $y \in Y$ and $x - y \in Z$, we have $Px = y$. This shows that P is a *closed* operator; the *closed graph theorem* tells us that P is, in fact, continuous ([L], 10.3). This operator P is called the projection from X on Y along Z.

On the other hand, starting with a projection operator $P \in BL(X)$ we obtain a decomposition of X as follows: Let $Y = R(P)$ and $Z = Z(P)$. Since P is continuous, Z is closed; also, since $Y = Z(I-P)$, where $I - P$ is continuous, we see that Y is closed. Moreover, for every $x \in X$, we have $x = Px + (x-Px)$, so that $X = Y + Z$. Clearly, $x \in Y \cap Z$ implies $x = Px = 0$. Thus,

$$X = R(P) \oplus Z(P) .$$