1.10. Comparison of Semigroups.

In perturbation theory one starts from a semigroup S and an operator P, which is "small" with respect to the generator H of S, and then constructs a perturbed semigroup S^P, with generator $H + P$, which is "close" to S. The notions of "smallness" of the perturbation and "closeness" of the semigroups are intimately related. In particular one can estimate from the identity

$$S_t - S_t^P = \int_0^t ds \frac{d}{ds} \left(S_{t-s}^P S_s\right)$$

$$= \int_0^t ds\, S_{t-s}^P P\, S_s$$

that

$$\|S_t - S_t^P\| = O(t) ,$$

as $t \to 0$, if P is bounded, or

$$\|(S_t - S_t^P)a\| = O(t)$$

for all $a \in D(H)$, as $t \to 0$, if P is relatively bounded with respect to H. Our aim is to prove converses to these statements.

We now begin with two semigroups satisfying the estimate (*), or (**), and attempt to prove that the corresponding generators differ by a bounded, or a relatively bounded, pertúrbation. The difficulty is that these converse statements are not valid for general C_0-semigroups. Nevertheless they are valid for C_0^*-semigroups,