1.9. Perturbation Theory

The next aspect of stability that we describe is stability of a semigroup under perturbations of its generator. Let H be the generator of a C_0-semigroup of contractions on the Banach space $\mathcal{B}$ and P a linear operator on $\mathcal{B}$. Our aim is to describe conditions on P which ensure that $H + P$ also generates a C_0-semigroup of contractions. In applications the perturbation P is often an unbounded operator and the notion of relatively bounded operator is useful.

Let H and P be linear operators on a Banach space. Then P is defined to be *relatively bounded* with respect to H, or H-*relatively bounded,* if the following two conditions are satisfied:

1. $D(P) \supseteq D(H)$

2. $\|Pa\| \leq \alpha\|a\| + \beta\|Ha\|$

for all $a \in D(H)$ and some $\alpha, \beta > 0$.

The greatest lower bound of the β for which this last relation is valid is called the *relative bound* of P with respect to H, or the H-*bound.*

The key result concerning relative bounded perturbations of generators of contraction semigroups is the following:

THEOREM 1.9.1. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *and assume* P *is* H-*relatively bounded with*