1.8. Convergence of Semigroups

In the preceding sections we examined the existence and construction of various classes of semigroup and next we analyze their stability properties. First we consider convergence properties and use these to extend the foregoing results on semigroup construction.

Let $S^{(n)}$ be a sequence of C_0-semigroups on a Banach space $\mathcal{B}$ and assume that $S_t^{(n)}$ converges strongly to S_t , for each $t \geq 0$. Since the product of strongly convergent sequences is strongly convergent the S_t must satisfy the semigroup property $S_s S_t = S_{s+t}$ for all $s, t \geq 0$, and of course $S_0 = I$. Nevertheless $S = \{S_t\}_{t\geq 0}$ is not necessarily a C_0-semigroup because of a possible lack of continuity. The simplest example of this phenomenon is given by the numerical semigroups $S_t^{(n)} = e^{-nt}$ acting on $\mathbb{C}$. The limit S satisfies $S_0 = I$, and $S_t = 0$ if $t > 0$; it is clearly discontinuous. Thus it is of interest to establish conditions for stability of C_0-semigroups under strong convergence and to identify stability criteria in terms of the generators.

Although the strong limit S of the sequence $S^{(n)}$ often fails to be a C_0-semigroup on the whole Banach space $\mathcal{B}$ it is possible that its restriction to a Banach subspace $\mathcal{B}_0$ is a C_0-semigroup. For example if $\mathcal{B} = \mathcal{B}_0 \oplus \mathbb{C}$ and $S_t^{(n)} = T_t \oplus e^{-nt}$, where T is a fixed C_0-semigroup on $\mathcal{B}_0$, then the limit S is discontinuous for a rather trivial reason; on the subspace $\mathcal{B}_0$ one has continuity, because $S = T$, and the discontinuity only occurs in the extra