1.7. Holomorphic Semigroups.

Among the many semigroups which occur in applications one class is very common, the holomorphic semigroups. Roughly speaking these are the semigroups $t \geq 0 \mapsto S_t \in \mathcal{L}(\mathcal{B})$ which can be continued holomorphically into a sector of the complex plane containing the positive axis. Among these semigroups one can also identify a subclass analogous to the M-bounded semigroups, i.e., the semigroups satisfying a bound of the form $\|S_t\| \leq M$. This subclass consists of holomorphic semigroups which are uniformly bounded within appropriate subsectors of the sector of holomorphy. For example if H is a positive self-adjoint operator on the Hilbert space $\mathcal{H}$ and $S_t = \exp\{-tH\}$ is the corresponding semigroup then $a \in \mathcal{H} \mapsto S_t a \in \mathcal{H}$ extends to a vector valued function holomorphic in the right half plane satisfying

$$\|S_z a\| = \|S_{\mathrm{Re}\ z} a\| \leq \|a\|$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}\ z \geq 0$. Thus S is a bounded holomorphic semigroup with the right half plane as region of holomorphy.

The general definition of these semigroups is as follows.

DEFINITION 1.7.1. *A* C_0*-semigroup* S *on the Banach space* $\mathcal{B}$ *is called a holomorphic semigroup if for some* $\theta \in \langle 0, \pi/2]$ *one has the following properties:*

1. $t \geq 0 \mapsto S_t$ *is the restriction to the positive real axis of a holomorphic operator function*