1.5. C_0^*-semigroups.

If the Banach space $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$, the pre-dual of $\mathcal{B}$, then it is of interest to study families of bounded operators $S = \{S_t\}_{t\geq 0}$ with the semigroup property $S_s S_t = S_{s+t}$ which are weak*-continuous in the sense that

1. $$\lim_{t\to 0+} (S_t f, a) = (f, a)$$

   for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$,

2. $$\lim_{\alpha} (S_t f_\alpha, a) = (S_t f, a)$$

   for all $t > 0$, all $a \in \mathcal{B}_+$, and all families f_α such that

   $$\lim_{\alpha} (f_\alpha, a) = (f, a) .$$

Such families are called C_0^*-semigroups. The simplest example is translations on $L^\infty(\mathbb{R})$ which has pre-dual $L^1(\mathbb{R})$.

Our first aim is to show that if S is a C_0^*-semigroup there exists an adjoint semigroup S_* on $\mathcal{B}_*$ such that

$$(S_t f, a) = (f, S_{*t} a) .$$

The weak*-continuity of S then implies the weak, and hence strong, continuity of S_*, i.e., the C_0^*-semigroup S is the adjoint of a C_0-semigroup S_*. This explains the name C_0^*-semigroup. In the sequel we demonstrate that much of the foregoing theory of C_0-semigroups can be carried over to the C_0^*-semigroups by duality