CHAPTER 5

HARMONIC MAPS BETWEEN SURFACES

5.1 NONEXISTENCE RESULTS

In this chapter, we want to present the existence theory for harmonic maps between closed surfaces, possibly with boundary. In the two-dimensional case, the regularity theory for minimizing maps is very easy, and the local geometry of the image does not lead to any difficulties in contrast to the situation we encountered in chapter 4 (cf. the example in section 4.1). This allows us to investigate in more detail what obstructions for the existence of harmonic maps are caused by the global topology of the image.

We first want to show some instructive nonexistence results which illustrate the difficulties we shall encounter later on when we try to prove existence results by variational methods.

Lemaire [L1] showed

PROPOSITION 5.1.1 *There is no nonconstant harmonic map from the unit disc* D *onto* S^2 *mapping* ∂D *onto a single point.*

Proof Suppose $u : D \to S^2$ is harmonic with $u(\partial D) = p \in S^2$. Since the boundary values of u are constant, u is also a critical point with respect to variations $u \circ \psi$, where $\psi : D \to D$ is a diffeomorphism, mapping ∂D onto itself, but not necessarily being the identity on ∂D.

Thus, one can use a standard argument to show that u is a conformal map (cf. [L1] or [M3], pp.369-372). Since u is constant on ∂D one can extend it by reflection as a conformal map on the whole of $\mathbb{R}^2$. But then this conformal map is constant on a curve interior to its domain of