CHAPTER 3

THE HEAT FLOW METHOD

Existence, regularity, and uniqueness results for a nonpositively curved image

3.1 APPROACHES TO THE EXISTENCE AND REGULARITY QUESTION

There are four different approaches to the existence and regularity theory of harmonic maps available. The first one is the so-called heat flow method. In order to find a harmonic map homotopic to a given map $g : X \to Y$, one investigates the parabolic system

$$(3.1.1) \qquad \begin{aligned} \frac{\partial f(x,t)}{\partial t} &= \tau(f(x,t)) && \text{for } x \in X \text{ and } t \geq 0 \\ f(x,0) &= g(x) && \text{for } x \in X \end{aligned}$$

and one tries to prove that a solution of (3.1.1) exists for all $t \geq 0$ and that $f(\cdot,t)$ converges to a harmonic map f as $t \to \infty$. That means one tries to deform g into a homotopic harmonic map by an analogue of heat dispersion on manifolds. One should compare this method with the gradient flow descent method common in Morse theory. Whereas this method in our case would lead to an ordinary differential equation for a mapping from X into the Sobolev space $W_2^1(X,Y)$, i.e. an infinite dimensional target space, and follow the gradient lines of the energy functional, the heat flow method instead leads to a partial differential equation for a mapping from X into the finite dimensional manifold Y.

The second approach tries to establish regularity (and a-priori estimates) for weak solutions f of the elliptic system

$$(3.1.2) \qquad \int_X \{\gamma^{\alpha\beta}(x)\, g_{ij}(f(x)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial \phi^j}{\partial x^\beta} - \gamma^{\alpha\beta}(x)\, \Gamma^i_{jk}(f(x)) \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \phi^i\}\, dX$$

$$\text{for all } \phi \in W_2^1 \cap L^\infty .$$