CHAPTER 2

GEOMETRIC PRELIMINARIES

Almost linear functions, approximate fundamental solutions, and representation formulae. Harmonic coordinates.

2.1 OUTLINE OF THE CHAPTER

This chapter begins with a collection of basic estimates for Jacobi fields and some convexity results. We mostly follow the elegant presentation in [BK].

We then introduce the notion of almost linear functions on a manifold, the main technical innovation of [JK1]. Whereas standard coordinate functions, e.g. Riemannian normal coordinates, have only rather poor regularity properties (cf. the example in 2.8) due to the fact that they involve not only the distance function but also angular terms, almost linear functions will be constructed by only using the distance function, which admits a sufficient control through Jacobi field estimates. The basic idea is to use the Euclidean identity $2\langle x, p-q\rangle = |x-q|^2 - |x-p|^2$ $(p = -q)$ as a definition. These functions satisfy almost, i.e. up to a small error term, the usual characterizations of linear functions in Euclidean space, e.g. that the first derivatives are constant, the second ones vanish, or the Taylor expansion terminates after the second term. These error terms are inevitable due to the presence of curvature, conceptually considered as a measure of deviation from Euclidean space. Such error terms, however, generally are of lower order than the other terms which appear already in the Euclidean versions of the formulae and hence can be easily absorbed. In particular, we discuss approximate fundamental solutions of the Laplace and heat equation on manifolds and derive representation formulae. Almost linear functions permit to gain one order of differentiation in such formulae by enabling us to also approximate the