RIESZ CAPACITY AND THE APPROXIMATION OF SOBOLEV FUNCTIONS BY SMOOTH FUNCTIONS

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This work, done jointly with William P. Ziemer [2], is a generalisation of the following result, proved by F.C. Liu in 1977 [1].

If Ω is a strongly Lipschitz domain in R^n , $f \in W^{\ell,p}(\Omega)$ (where $1 \leq p < \infty$) and $\varepsilon > 0$ is arbitrary, then there exists a C^{ℓ} function g on Ω such that

(i) the set $\{x \, ; \, x \in \Omega \text{ and } f(x) \neq g(x)\}$ has Lebesgue measure $< \varepsilon$

and (ii) $|f-g|_{\ell,p} < \varepsilon$.

The Michael-Ziemer generalisation gives a C^m function g on Ω (with $m \leq \ell$) , the approximation in (i) is with respect to capacity and the approximation in (ii) is with respect to the norm in $W^{m,p}$. Moreover, Ω is an arbitrary open subset of R^n .

Riesz capacity can be defined in the following way. Let $1 \leq p < \infty$ and let k be a real number, such that $k > 0$ and $kp < n$. Let $f \in L^p(R^n)$ and suppose $f \geq 0$. The Riesz potential $I_k f$ of f is the function defined on R^n by

$$I_k f(x) = \frac{1}{\gamma(k)} \int_{R^n} |x-y|^{k-n} f(y)\,dy ,$$

where $\gamma(k)$ is a positive constant whose value is not important in the present context. For each subset E of R^n , the Riesz capacity $R_{k,p}(E)$ is defined by