THE EXISTENCE OF MAXIMAL SURFACES

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The minimal surface (Plateau) problem is well-known - one seeks a surface with minimal area amongst all surfaces spanning a given boundary. Instead we ask the analogous question in a Lorentzian space, so in the simplest case we are considering spacelike surfaces in flat Minkowski space $\mathbb{R}^{3.1}$ which maximise area. Recall that $\mathbb{R}^{3.1}$ is the 4-dimensional Euclidean space with metric $\sum_1^3 dx^{i^2} - dt^2$ and that a vector (x,t) is spacelike/timelike/null according as $|x|^2 - t^2 > 0 \;/\; < 0 \;/\; = 0$ respectively. A surface $M = \text{graph}_\Omega u$, $u \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^3$ is spacelike if all its tangent vectors are spacelike. This means that the induced metric g_{ij} is Riemannian,

(1) $$g_{ij} = \delta_{ij} - u_i u_j > 0,$$

where $u_i = \dfrac{\partial u}{\partial x^i}$, and hence $|Du| < 1$. The maximal surface equation is the Euler-Lagrange equation arising from the induced area functional:

(2) $$\text{Area}(M) = \int_\Omega \sqrt{\det g_{ij}}\, dx = \int_\Omega \sqrt{1-|Du|^2}\, dx\,,$$

and in Minkowski space can be written

(3) $$\frac{1}{\sqrt{1-|Du|^2}} \left(\delta_{ij} - \frac{D_i u\, D_j u}{1-|Du|^2} \right) D^2_{ij} u = 0.$$

Like the minimal surface equation, this is a nonlinear, non-uniformly elliptic equation and apriori estimates for $|Du|$ are needed