SOME REMARKS ON NUMERICAL METHODS FOR NONLINEAR HEAT EQUATIONS WITH NEAR SINGULAR SPECIFIC HEATS

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Consider the one-dimensional nonlinear heat conduction equation

$$(1)\qquad \begin{cases} c(u)\,\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left[k(u)\,\dfrac{\partial u}{\partial x}\right] + q(x,t)\,, & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0\,, & t > 0 \\ u(x,0) = u_0(x)\,, & x \in (0,1)\,, \end{cases}$$

where $c(\cdot)$ and $k(\cdot)$ are continuous functions on $\mathbb{R}$ satisfying

$$(2a)\qquad \exists c^* > 0 \text{ such that } c(u) \geq c^* > 0 \quad \forall u \in \mathbb{R}$$

$$(2b)\qquad k(u) > 0 \quad \forall u \in \mathbb{R}\,.$$

Of particular interest are cases where $c(u)$ varies greatly over a small temperature range. Such behaviour can arise in simple models of phase changes in alloys ("near Stefan problems"). It would be desirable to have a numerical method for approximating the solution of (1) whose accuracy was in some sense independent of the behaviour of the coefficients $c(\cdot)$ and $k(\cdot)$. It is however unreasonable to expect this much since the accuracy of any approximation will clearly be influenced by the regularity of the exact solution. This regularity can vary considerably, depending on the coefficient $c(\cdot)$, $k(\cdot)$ as well as the initial temperature data u_0 and the source data q. A more reasonable request would be that the stability properties of a numerical method be uniform for all $c(\cdot)$ and $k(\cdot)$ satisfying (2). That is, we would like to be able to assert something of the form: