$W^{2,p}$ REGULARITY FOR VARIFOLDS WITH MEAN CURVATURE

John Duggan

Suppose $V = \underline{v}(M,\tilde{\theta})$ is a rectifiable n-varifold in $\mathbb{R}^{n+k}$ with generalised mean curvature $\underline{\underline{H}} \in L^p(\mu, \mathbb{R}^{n+k})$ in U, that is

(1) $$\int \operatorname{div}_M X \, d\mu = -\int X.\underline{\underline{H}} \, d\mu$$

for all $X \in C_0^1(U, \mathbb{R}^{n+k})$, where $\mu = \mathcal{H}^n \llcorner \tilde{\theta}$. Then, if $\tilde{\theta} \geq 1$ μ-ae in U, $p > n$, $0 \in \operatorname{spt}\mu$ and $B_\rho(0) \subset U$, the regularity theorem ([A]) states that there are $\gamma = \gamma(n,k,p)$, $\delta = \delta(n,k,p) \in (0,1)$ such that

$$\frac{\mu(B_\rho(0))}{\omega_n \rho^n} \leq 1 + \delta , \qquad \left(\int_U |\underline{\underline{H}}|^p \, d\mu\right)^{1/p} \rho^{1-n/p} \leq \delta$$

imply that $\operatorname{spt}\mu \cap B_{\gamma\rho}(0) = q(\text{graph } u) \cap B_{\gamma\rho}(0)$ for some linear isometry q of $\mathbb{R}^{n+k}$ and some $u \in C^{1,1-n/p}(B_{\gamma\rho}(0), \mathbb{R}^k)$. (Here $B^n_{\gamma\rho}(0) = B_{\gamma\rho}(0) \cap \mathbb{R}^n$.) We show here that a higher regularity prevails, and that u is actually $W^{2,p}$ and that the density function $\tilde{\theta}$ is $W^{1,p}$.

We write (1) in non-parametric form:

(2) $$\begin{cases} \displaystyle\int \theta \sqrt{g}\, g^{i\ell} D_\ell \eta = -\int \theta \sqrt{g}\, H^i \eta , & 1 \leq i \leq n \\ \displaystyle\int \theta \sqrt{g}\, g^{m\ell} D_m u^j D_\ell \eta = -\int \theta \sqrt{g}\, H^{n+j} \eta , & 1 \leq j \leq k , \end{cases}$$

for $\eta \in C_0^1(\Omega)$, Ω a domain in $\mathbb{R}^n$. Because the results we obtain hold quite generally, as well as in order to simplify the exposition, we consider, instead of (2), the following system:

(3) $$\int_\Omega \theta\, \Phi_i^s(Du)\, D_i \eta = -\int_\Omega \theta H^s \eta , \qquad 1 \leq s \leq n+k .$$

Then we have