HARMONIC MORPHISMS ONTO RIEMANN SURFACES — SOME CLASSIFICATION RESULTS

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1. INTRODUCTION

Let $\phi : M \to N$ be a mapping between smooth Riemannian manifolds. Then ϕ is called a *harmonic morphism* if $f \circ \phi$ is harmonic on $\phi^{-1}(V)$ for every function f harmonic on an open set $V \subset N$. Such mappings were first studied in detail by Fuglede [8] and Ishihara [10]. They established an alternative characterization as follows.

For a point $x \in M$ at which $d\phi(x) \neq 0$, let V_xM denote the subspace of T_xM given by $\ker d\phi(x)$, and let H_xM denote the orthogonal complement of V_xM in T_xM. Say that ϕ is *horizontally conformal* if the restriction mapping $d\phi(x)\big|_{H_xM} : H_xM \to T_{\phi(x)}N$ is conformal and surjective. Letting g,h denote the metrics of M,N respectively, this means that there exists a number $\lambda(x)$ such that $\lambda(x)^2\, g(X,Y) = h(d\phi(X), d\phi(Y))$ for each $x \in M$ with $d\phi(x) \neq 0$ and for all $X,Y \in H_xM$. Let $C_\phi = \{x \in M \mid d\phi(x) = 0\}$ denote the *critical set of* ϕ, and set $\lambda = 0$ on C_ϕ. Then we obtain a continuous function $\lambda : M \to \mathbb{R}$ called the *dilation of* ϕ. In general λ is not smooth, although clearly $\lambda^2 : M \to \mathbb{R}$ is a smooth function.

(1.1) *A map* $\phi : M \to N$ *is a harmonic morphism if and only if it is both harmonic and horizontally comformal* [8], [10].

It follows therefore that if ϕ is a harmonic morphism then $\dim M \geq \dim N$. If $\dim M = \dim N$, then in the case when $\dim M = 2$, the harmonic morphisms are precisely the weakly conformal mappings between