A SUPPORT MAP CHARACTERIZATION OF THE OPIAL CONDITIONS

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A Banach space [dual space] X satisfies the *weak* [*weak**] *Opial condition* if whenever (x_n) converges weakly [weak*] to x_∞ and $x_0 \neq x_\infty$ we have

$$\lim_n \inf \|x_n - x_\infty\| < \lim_n \inf \|x_n - x_0\|.$$

Zdzisław Opial [1967] introduced the weak condition to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a nonexpansive selfmapping of a closed convex subset to a fixed point. In particular he observed that a uniformly convex Banach space with a weak to weak* sequentially continuous support mapping satisfies the weak condition. A *support mapping* is a selector for the *duality map*

$$D: X \to 2^{X^*} : x \mapsto \{f \in X^*: f(x) = \|f\|^2 = \|x\|^2\}$$

Uniform convexity is not sufficient for the weak to weak* sequential continuity of the unique support mapping. Browder [1966], and independently Hayes and Sims in connection with operator numerical ranges, had observed that the uniformly convex space $L_4[0, 1]$ does not have a weak to weak (= weak*) continuous support mapping, while all of the sequence spaces ℓ_p $(1 < p < \infty)$ do. Opial [1967] demonstrated that with the exception of $p = 2$ none of the spaces $L_p[0, 1]$ have weak to weak continuous support mappings. Indeed, Fixman and Rao characterize $L_p(\Omega, \Sigma, \mu)$ spaces with a weak to weak continuous support mapping as those spaces for which every element of Σ with finite positive measure contains an atom.

That uniform convexity is not necessary is shown by the example of ℓ_1 with an equivalent smooth dual norm. That the unique support mapping is