SOME APPLICATIONS OF HARMONIC ANALYSIS TO NUMBER THEORY

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Two sets A,B of real numbers, all exceeding unity, are termed *multiplicatively independent* if no relation of the form $r^m = s^n$ holds for $m,n \in Z^+$, $r \in A$, $s \in B$. We define a real number x to be *normal to base* r if the sequence $(r^n x)_{n=0}^{\infty}$ is uniformly distributed modulo unity (*cf.* Mendès France [12]). In the case of an integer base this definition has been shown by Wall [21] (see also Niven [13]) to be equivalent to the more usual definition involving asymptotic frequencies of all possible digit blocks.

It has long been known that almost all numbers (in the sense of Lebesgue measure λ) are normal with respect to any given base (Weyl [22]). Schmidt [18] has established that if integers r,s are multiplicatively dependent, then normality to base r entails normality to base s. The proof of Kuipers and Neiderreiter ([11], Theorem 8.2) covers the non-integer case. The question of what class of numbers may be simultaneously normal to one base and non-normal to another is more subtle. Stimulus for research in this area has largely stemmed from work of Cassels [8] and Schmidt [18]. Schmidt was the first to prove in full generality that if r and s are multiplicatively independent integers, then the set of numbers normal to base r but not to base s is uncountable. His proof, which is fairly intricate, was based on use of the support set of the general Cantor measure on [0,1] with constant integer ratio of dissection. Shortly afterwards Schmidt [19] showed that if A,B