BASIC MEASURES

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1. INTRODUCTION

We are all familiar with convolution as a smoothing operation. An example of this is the classical theorem of Steinhaus that

(1.1) $$|E| > 0 \Rightarrow E + E \text{ contains an interval.}$$

One simple way of proving (1.1) is to consider the convolution of the indicator function of E with itself.

Now let C denote Cantor's middle third set and let μ_c be a probability measure evenly distributed over C. Since $|C| = 0$, it is obvious that

(1.2) $$\mu_c \perp \lambda \quad \text{(where } \lambda \text{ denotes Lebesgue measure).}$$

Less obviously

$$\mu_c * \mu_c \perp \lambda,$$

despite the fact that $C + C$ fills out an interval. Indeed some support sets of convolution powers of μ_c must be quite small because

(1.4) $$\mu_c^n \perp \mu_c^m \perp \lambda, \quad n \neq m.$$