OVERDETERMINED SYSTEMS DEFINED BY COMPLEX VECTOR FIELDS

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§1. FORMALLY INTEGRABLE STRUCTURE

Let Ω be a C^∞ manifold (Hausdorff, countable at infinity), $\dim \Omega = N(\geq 1)$, and let $L_L, \ldots, L_n$ be n complex vector fields, of class C^∞, in Ω, linearly independent at every point (so that $n \leq N$). We would like to study the *homogeneous* equations

$$L_j h = 0 \ , \ j = 1, \ldots, n \ , \tag{1}$$

as well as the *inhomogeneous* equations

$$L_j u = f_j \ , \ j = 1, \ldots, n \ , \tag{2}$$

with right-hand sides $f_j \in C^\infty(\Omega)$. It is known from the study of a single vector field (*i.e.*, $n = 1$) that difficulties arise even at the *local* level. In this expository note I shall limit myself to the local viewpoint and Ω can be taken to be an open subset of Euclidean space $\mathbb{R}^N$. Yet it is perhaps advisable to continue thinking of Ω as a manifold lest the important consideration of invariance be forgotten.

The questions one begins by asking, about equations (1) and (2), are the standard ones: existence, uniqueness and approximation of solutions, their regularity, their representations (say, by means of integral operators), etc. Answering these questions with satisfactory generality seems to be very difficult. Here I shall briefly describe