EXISTENCE OF WILLMORE SURFACES

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For compact surfaces Σ embedded in $\mathbb{R}^n$, the Willmore functional is defined by

$$F(\Sigma) = \frac{1}{2}\int_\Sigma |\underline{\underline{H}}^2|$$

where the integration is with respect to ordinary 2-dimensional area measure, and H is the mean curvature vector of Σ (in case $n = 3$ we have $|\underline{\underline{H}}| = |\kappa_1 + \kappa_2|$, where κ_1, κ_2 are principal curvatures of Σ). In particular $F(S^2) = 8\pi$.

For surfaces Σ without boundary we have the important fact that $F(\Sigma)$ is invariant under conformal transformations of $\mathbb{R}^n$; thus if $\tilde{\Sigma} \subset \mathbb{R}^n$ is the image of Σ under an isometry or a scaling ($x \mapsto \lambda x$, $\lambda > 0$) or an inversion in a sphere with centre not in Σ (e.g. $x \mapsto x/|x|^2$ if $0 \notin \Sigma$) then

(1) $$F(\Sigma) = F(\tilde{\Sigma}) .$$

(See [WJ], [LY], [W] for general discussion.)

For each genus $g = 0, 1, 2, \ldots$ and each $n \geq 3$ we let

$$\beta_g^n = \inf F(\Sigma) ,$$

where the inf is taken over compact genus g surfaces without