ON THE STRUCTURE OF BRANCH POINTS OF MINIMIZING DISKS

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§1. Introduction and statement of results.

Let $\mathbf{B}$ be the closed unit disk in $\mathbf{C}$. A smooth map F of $\mathbf{B}$ into an n-dimensional manifold N is said to have a <u>branch point of order $Q-1$</u> at $0 \in \mathbf{B}$ if there exist local co-ordinates $(x_1, \ldots, x_n)$ of a neighborhood of $F(0)$ with respect to which F takes the form

$$
\begin{aligned}
x_1 + \sqrt{-1}\, x_2 &= z^Q + o(|\, z\,|^Q) \\
x_k &= o(|\, z\,|^Q), \qquad 3 \leq k \leq n,
\end{aligned}
$$

where Q is an integer ≥ 2.

Branch points fall into two categories, true and false. A branch point of order $Q-1$ at $0 \in \mathbf{B}$ is <u>false</u> if there exists an immersion $\tilde{F} : \mathbf{B} \to N$ and $\psi : \mathbf{B} \to \mathbf{B}$ of degree Q with $\psi(0) = 0$ such that $F = \tilde{F} \circ \psi$. A branch point is <u>true</u> if it is not false. Thus, the image of a map with a false branch point is a smooth submanifold of N, whereas the image of a map with a true branch point is singular in the usual sense of differential geometry.

Branch points arise very naturally in the theory of minimal surfaces. They are the simplest type of singularity that a minimal surface could possess. A recent spectacular result of Sheldon Chang [C] shows that in fact they are the <u>only</u> possible singularities of area-minimizing two dimensional integral currents. For a history of the study of branch points we refer to [02]. The results most closely related to the ones in this article are the following.

THEOREM 1 *(Osserman [01]) Let $F : \mathbf{B} \to \mathbf{R}^3$ define a minimal surface which has a true branch point at 0. Then, given an arbitrarily small neighborhood V of 0, there exists a piecewise smooth map $\overline{F} : \mathbf{B} \to \mathbf{R}^3$ which agrees with F on $\mathbf{B} \backslash V$ and such that Area $(\overline{F}(V)) <$ Area $(F(V))$.*