CAPILLARY SURFACE REGULARITY IN CORNER SUBDOMAINS OF R^n

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The nonparametric capillary problem is to find a surface S_u=graph(u) above a subdomain Ω of R^n so that S_u has prescribed mean curvature above Ω and makes prescribed angle of contact with the bounding cylinder $\Sigma=\partial\Omega \times R$. Letting υ be the downward normal to S_u (or its first n components when appropriate), and letting γ be the inner normal to Σ, this quasilinear elliptic boundary value problem can be written as

$$\text{CP} \qquad \begin{array}{ll} \operatorname{div} \upsilon = \Psi & \text{in } \Omega, \text{ where } \Psi_u \geq 0. \\ \upsilon \cdot \gamma = \Phi & \text{on } S_u \cap \Sigma, \text{ where } \Phi_u \geq 0 \text{ and } |\Phi| < 1-\delta. \end{array}$$

The capillary problem has been solved both variationally (using functions of bounded variation or geometric measure theory), and by using an elliptic partial differential equation approach that combines apriori estimates with the method of continuity. For smooth domains the solution u exists and is regular on the closed domain, at least in the case that one can prove an a priori height estimate $|u| \leq M$. (This is always the case if gravity is positive, $\Psi_u \geq \delta > 0$, but may not be the case in general. Without the assumtion of positive gravity the shape of Ω becomes important.)

The capillary problem makes sense even if $\partial\Omega$ has a compact (n-1)-dimensional singular set Γ. (The variational problem can still be solved, or alternately the P.D.E. approach can be combined with a domain approximation argument, to find a function that solves CP everywhere except on Γ.) In this case, at least for positive gravity, one knows that the solution is smooth away from Γ, and it is natural to study its behavior near Γ. For two-dimensional corner domains, where Γ is a point at which Ω has an interior angle θ, and where the contact angle is ϕ (i.e. $\upsilon \cdot (-\gamma) = \cos\phi$) the somewhat surprising results have been known for several years [1][6][2]:

(a) If $\theta < |\pi - 2\phi|$ the solution to CP is either unbounded at Γ or it doesn't exist (depending on whether gravity is positive or not).