# BERNSTEIN THEOREMS FOR HARMONIC MORPHISMS

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## 0. INTRODUCTION

Let $\phi : M \to N$ be a continuous mapping between connected smooth Riemannian manifolds. Then $\phi$ is called a *harmonic morphism* if for every function f harmonic on an open set $V \subset N$, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subset M$. It follows by choosing smooth harmonic local coordinates on N [11], that any harmonic morphism is necessarily smooth.

The harmonic morphisms are precisely the harmonic maps which are *horizontally weakly conformal* (see [10], [14]). For a map $\phi : \mathbf{R}^3 \to \mathbf{C}$ this is equivalent to $\phi$ satisfying the equations

$$\sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \qquad (0.1)$$

$$\sum_{i=1}^{3} \left(\frac{\partial \phi}{\partial x_i}\right)^2 = 0 \ . \qquad (0.2)$$

Harmonic morphisms are a subject of considerable interest. Their history goes back to Jacobi [15] in 1847, who considered the problem of finding compex-valued (harmonic) functions $\phi$ on $\mathbf{R}^3$ satisfying (0.1) and (0.2) above. More recently they have been studied in the context of stochastic processes, where they are found to be the *Brownian path preserving mappings* (see [5]). In fact our main Theorem (Theorem 1) solves a problem first posed by Bernard, Cambell and Davie in [5].

The study of harmonic morphisms from domains in $\mathbf{R}^m$ to a Riemann surface has striking analogies with the study of minimal surfaces in $\mathbf{R}^m$. For example, the fibres of such a harmonic morphism are minimal and the associated Gauss map (see [4] for definition) obeys a holomorphicity condition similar to that for a minimal immersion (see[6]). In fact the analogies are so striking that on expects to find corresponding results more generally. This turns out to be true for the well known Bernstein Theorems for complete minimal surfaces in $\mathbf{R}^3$ (see [16]). We show that the only non-constant harmonic morphism $\phi$ defined on the whole of $\mathbf{R}^3$, taking values in a Riemann surface N, is the simplest possible, namely an orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ followed by a weakly conformal mapping $\mathbf{R}^2 \to N$.

Similarly for $S^3$, where the only non-constant harmonic morphism $\phi : S^3 \to N$ taking