BANACH SPACES WITH MANY PROJECTIONS

David Yost and Brailey Sims

A fundamental question in operator theory is: how rich is the collection of operators on a given Banach space? For classical spaces, especially Hilbert space, a well developed theory of operators exists. However a Banach space may have far fewer operators than one might expect. Shelah [S] has constructed a nonseparable Banach space X, for which the space of operators with separable range has codimension one in $B(X)$. An interesting open problem is whether there is a Banach space X for which the space of compact operators has finite codimension in $B(X)$.

For many classical Banach spaces, the projections generate $B(X)$. For example, every operator on a Hilbert space is a linear combination of at most ten self-adjoint projections [M]. This is false for ℓ_∞, since every projection thereon, unless of finite rank, has nonseparable range [L2]. It is still unknown whether every Banach space admits a nontrivial projection. (By nontrivial we mean that both the range and the null space are infinite dimensional.) In this report we will be interested in a particular type of projection.

Let X be a Banach space and let M be a closed subspace of X. By a linear extension operator (LEO) we mean a linear mapping $T : M^* \to X^*$ such that, for each f in M^*, Tf is a norm-preserving extension of f. A routine exercise shows that there exists a LEO from M^* to X^* if, and only if, M^o is the kernel of a contractive projection on X^*. Most subspaces of most Banach spaces admit no LEO. To obtain reasonable results we must restrict our attention to a smaller class of Banach spaces.

An Asplund space is a Banach space for which every separable subspace has a separable dual. This is equivalent to every subspace having the same density character as its dual [Ph]. We use this property to show that every non-separable Asplund space has many subspaces (in fact, an uncountable increasing family of them) which admit LEOs. Hence the dual of an Asplund space (that is, a dual space with the Radon-Nikodym Property [Ph]) admits many projections. Moreover these projections can be chosen to be well behaved in a certain sense - in particular they all commute.

In the second section we consider the problem of renorming Asplund spaces and their duals. The existence of many projections is a useful tool for this, and other problems. It is well known that any space with an equivalent Frechet smooth norm is automatically an Asplund space. The converse remains open. Examples, [E] and [Tl], show that it may not be possible to renorm an Asplund space so that the dual norm is reasonably convex. We conjecture that the dual of every Asplund space can be renormed so as to be locally uniformly convex, although the known proofs all require some additional smoothness hypothesis. Such renormings need not of course be dual renormings.