OPERATORS WHICH HAVE AN H_∞ FUNCTIONAL CALCULUS

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1. INTRODUCTION

An operator T in a Hilbert space $\mathcal{H}$ is said to be of type ω if the spectrum is contained in the sector $S_\omega = \{\zeta \in \mathbb{C} \mid |\arg\zeta| \le \omega\}$ and the resolvent satisfies a bound of the type $\|(T-\zeta I)^{-1}\| \le C_\mu |\zeta|^{-1}$ for all ζ with $|\arg\zeta| \ge \mu$ and all $\mu > \omega$. Let us suppose for now that T is one-one with dense range.

Such an operator has a fractional power T^s and, if $\omega < \pi/2$, generates an analytic semi-group $\{\exp(-tT)\}$. See [3] for details. However it may or may not happen that it generates a C^0-group $\{T^{is} \mid s \in \mathbb{R}\}$ of bounded operators. It was shown by Yagi that the operators T for which $T^{is} \in \mathcal{L}(\mathcal{H})$ are precisely those for which the domains of the fractional powers of T (and of T^*) are the complex interpolation spaces between $\mathcal{H}$ and $\mathcal{D}(T)$ (and between $\mathcal{H}$ and $\mathcal{D}(T^*)$). They are also precisely those operators for which T and T^* satisfy quadratic estimates [4].

It is shown in this paper that another equivalent property is the existence of an $H_\infty(S_\mu^0)$ functional calculus for $\mu > \omega$ (where S_μ^0 denotes the interior of S_μ).

In writing up this paper it seemed useful to have a precise definition of the operators $f(T)$ for functions which are analytic (but not necessarily bounded) on S_μ^0 and for operators T which do not necessarily satisfy quadratic estimates. Such a definition is given in section 5, where it is shown in what sense formulae of the form $(fg)(T) = f(T)g(T)$ hold. It appears that the basic properties of the