SINGULAR INTEGRALS AND MAXIMAL FUNCTIONS ON CERTAIN LIE GROUPS

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INTRODUCTION

Let T be a convolution operator on $L^p(\mathcal{R}^n)$, $1 \le p < \infty$ whose operator norm is denoted $||T||_p$. Then two basic results in the calculus of convolution operators are these.

PROPOSITION. *(i) If* $1 \le p < \infty$ *and* T *is bounded on* $L^p(\mathcal{R}^n)$, *then* T *is also bounded on* $L^{p'}(\mathcal{R}^n)$, *and*

$$||T||_p = ||T||_{p'}.$$

(ii) If T *is bounded on* $L^p(\mathcal{R}^n)$, *then* T *is bounded also on* $L^2(\mathcal{R}^n)$, *and*

$$||T||_2 \le ||T||_p.$$

In fact, (ii) is a consequence of (i) because of interpolation.

Now if we pass to a general locally compact Hausdorff group G with left invariant Haar measure, we may consider a kernel K and the corresponding operator $T_K = K\star$:

$$T_K f(x) = K \star f(x) = \int_G K(y) f(y^{-1}x)\, dy.$$

It was proved some time ago that if $1 \le p \le 2$, and T_K is bounded on $L^p(G)$, and G is *amenable*, then T_K is bounded on $L^2(G)$, and

$$||T||_2 \le ||T||_p.$$

This is due to C. S. Herz [1].

Amenable groups share certain properties with abelian locally compact groups, viz. the possibility of constructing certain kinds of bounded local units. They include the compact, the nilpotent and the solvable Lie groups. However, noncompact semi-simple Lie groups such as $SL^2(\mathcal{R})$ are *not* amenable. This is borne out dramatically by the following result of N. Lohoué [2].

THEOREM. *Let* G *be a noncompact semi-simple Lie group with finite center. Suppose* $1 < p_0 < \infty$. *Then there is a positive measure* μ *on* G *which convolves* L^{p_0} *into* L^{p_0} *but does not map* $L^p(G)$*into* $L^p(G)$ *for any other* p.

Problem of asymmetry If G is a locally compact group, $1 \le p < \infty$, and K is a kernel on G such that T_K maps $L^p(G)$into $L^p(G)$, does K also convolve $L^{p'}(G)$into $L^{p'}(G)$? If it does, do we have

$$||T_K||_p = ||T_K||_{p'}.$$

Note that if G is a finite group, then the answer to the first question is in the affirmative. However, D. Oberlin [3] showed a number of years ago that if G is the dihedral group of order 8, then there exist kernels which have *different* convolution norms on the spaces $L^p(G)$and $L^{p'}(G)$. In case the answer to both questions above is negative, we say that the group is *asymmetric*. In case the answer to the first question is affirmative, and to the second negative, we say the group is *weakly asymmetric*

Nilpotent groups Consider the Heisenberg group H of dimension 3, $H \cong \mathcal{R}^2 \times \mathcal{R}$ in which the multiplication law is

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$