DERIVATIVES OF INVARIANT POLYNOMIALS ON A SEMISIMPLE LIE ALGEBRA

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0. INTRODUCTION.

Let $\underline{\underline{g}}$ be a complex semisimple Lie algebra and let G be the adjoint group of $\underline{\underline{g}}$. It is known that the algebra $I(\underline{\underline{g}}) = \mathbf{C}[\underline{\underline{g}}]^G$ of G-invariant polynomial functions on g is generated by $n = rank(\underline{\underline{g}})$ algebraically independent homogeneous polynomials $P_1, \ldots, P_n$. Let $\pi : \underline{\underline{g}} \to \mathbf{C}^n$ be the polynomial map given by $\pi(x) = (P_1(x), \ldots, P_n(x))$; π is the *quotient morphism* of $\underline{\underline{g}}$. The geometry of the quotient morphism has been studied in detail by Kostant [9] and his results have played an important role in many problems in representation theory. Among other results, he shows that the differential $d\pi_x$ at a point $x \in \underline{\underline{g}}$ is of maximal rank n if and only if x is a regular element of $\underline{\underline{g}}$. Our goal is to compute $\text{rank}(d\pi_x)$ for *every* $x \in L(G)$. We have succeeded except for the case of one nilpotent conjugacy class in type E_8.

It clearly suffices to handle the case in which $\underline{\underline{g}}$ is simple. By using the Luna slice theorem, we can reduce to the case when x is a nilpotent element of $\underline{\underline{g}}$. For the classical simple Lie algebras, we can give a reasonably straightforward computation of the ranks. For the exceptional Lie algebras, we need to use the classification of nilpotent conjugacy classes and detailed information on the closures of nilpotent classes.

It is convenient to reinterpret our results in terms of G-invariant (polynomial) vector fields on $\underline{\underline{g}}$. A G-invariant vector field on $\underline{\underline{g}}$ is just a G-morphism $\varphi : \underline{\underline{g}} \to \underline{\underline{g}}$. For each $j = 1, \ldots, n$, let $\varphi_j = \text{grad}(P_j)$. It is known that $\varphi_1, \ldots, \varphi_n$ are a basis for the $I(\underline{\underline{g}})$-module of G-invariant vector fields. Using this result, it is easy to show that the following three numbers are equal for every $x \in \underline{\underline{g}}$: (i) $\text{rank}(d\pi_x)$; (ii) the dimension of the vector subspace of $\underline{\underline{g}}$ spanned by $\{\varphi_1(x), \ldots, \varphi_n(x)\}$; and (iii) the multiplicity of the adjoint representation of G in the G-module $\mathbf{C}[\overline{G \cdot x}]$ of regular functions on $\overline{G \cdot x}$, the closure of the orbit $G \cdot x$.

By using the result above and a theorem of Borho and Kraft [3], we obtain a number of new examples of non-normal nilpotent orbit closures in the exceptional Lie algebras of types E_6, E_7 and E_8. For example, we show that (at least) seven of the twenty-one nilpotent orbits in type E_6 have closures which are not normal varieties.

This paper contains only a statement of results, with outlines of the proofs. A detailed exposition of these results will appear elsewhere.