The Laplace Beltrami Operator on Unbounded Homogeneous Domains in C^n

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Let Ω be a domain in C^n. A Lie group G is said to act on Ω if G act as a group and the mapping μ of $G \times \Omega$ into Ω is real analytic in G and holomorphic on Ω . The action is said to be rational if in addition it is rational on Ω.

For the remainder of this talk, we shall assume that G acts homogeneously, rationally, and that Ω carries a G invariant volume. Then Koszul [K] showed that Ω has a canonically defined two form – the Koszul form. Assume that this form is nondegenerate . Then Ω is a pseudo-Kahlerian domain. Hence Ω carries a canonical second order differential operator – the Laplace Beltrami operator. If Ω is not Kahlerian, then then this operator is not elliptic and not positive. However it is invariant under any holomorphic mapping that preserves the volume form. In particular, the spectrum of this operator (assuming that it is essentially self adjoint) is an invariant of the domain.

It is the goal of this talk to describe a method of computing the spectrum of this operator in a broad class of domains. To describe this class of domains, we shall need some structure theory (for this see [P2]). It turns out that all such domains are describable in terms of nilpotent Lie groups.

Let N be a nilpotent Lie group with Lie algebra $\mathcal{N}$. Let $\mathcal{N}_c$ be the complexification of $\mathcal{N}$ and let N_c be the corresponding connected, simply connected, Lie group. Let $\mathcal{P}$ be a complex sub algebra of $\mathcal{N}_c$. Let A denote the group of all automorphisms of $\mathcal{N}_c$ which leave both $\mathcal{N}$ and $\mathcal{P}$ invariant. Then A acts on both N_c and $X = N_c/P$. Let T be a maximal, R-split torus in the algebraic group A. We shall say that the pair $N - P$ is a Siegel $N - P$ pair if T_o (the connected component of T) has an open orbit in X. In this case there are only a finite number of open orbits. Each such orbit is a homogeneous domain for the group $S = T_o \times_s N$. Furthermore, due to the conjugacy of maximal tori, the domains do not depend on the choice of torus. We shall refer to these domains as the domains defined by the Siegel pair $N - P$. The basic structure theorem of [P2] is the following: