ADDITIVE SET FUNCTIONS OF BOUNDED Φ–VARIATION

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The notion of a (point) function of finite p–variation was introduced by N. Wiener in [5]. It was extended by L.C. Young who considered, in [6], functions of finite Φ–variation, where Φ is an increasing function on $[0,\infty)$. Young also gave sufficient conditions for the existence of the Stieltjes integral

$$\int_a^b f \mathrm{d} g \tag{0.1}$$

in terms of the Φ–variation of the function f and Ψ–variation of the function g in the interval $[a,b]$. Such integrals were from this point of view subsequently studied by several authors including Young himself. Using a very interesting idea, A. Beurling improved Young's condition in [1]. However, this idea does not seem easy to generalize so as to cover additive set functions in abstract spaces; it uses the fact that (0.1) is essentially integral with respect an additive set functions defined on sub–intervals of $[a,b]$. In this note, methods remotely akin to that of Beurling are presented for introducing and studying integrals with respect to set functions on semialgebras in abstract spaces. The interest in such enterprise stems from various, seemingly unrelated, sources: stochastic fields (processes with multidimensional time–parameter), spectral theory, Feynman integral and, possibly, others.

1. Let $\mathcal{Q}$ be a semialgebra of sets in a space Ω. That is, $\mathcal{Q}$ is a semiring (cf. [2],4.6) such that $\Omega \in \mathcal{Q}$. By a partition will be understood a finite family of pair–wise disjoint sets from $\mathcal{Q}$ whose union is equal to Ω. The set of all such partitions is denoted by Π. If the partition $\mathcal{P}'$ is a refinement of the partition $\mathcal{P}$, we write $\mathcal{P} \prec \mathcal{P}'$.

We shall abuse the notation by writing $f(X) = \{f(\omega)\colon \omega \in X\}$ for any function f on Ω and a set $X \subset \Omega$. Furthermore, the same symbol will be used to denote a