A Heat Equation Approach to Boutet de Monvel's Index Theorem for Toeplitz Operators

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§1. The index theorem on the circle

In this talk, we will show how heat-kernel methods can be used to prove Boutet de Monvel's index theorem for Toeplitz operators on a compact strictly pseudo-convex CR manifold. Since this theorem implies the Atiyah-Singer index theorem for an arbitrary elliptic pseudodifferential operator (the proof of this fact makes use of an explicit Fourier integral operator, see the appendix to [4]), this shows how to give a purely analytic proof of the Atiyah-Singer index theorem, without any use of K-theory. In fact, we will only give the first step in the argument, the proof of a McKean-Singer formula for Toeplitz operators; the actual calculation of the index will be found in [6].

Some of the ideas of the proof are already apparent on the simplest of CR manifolds, the circle (where Toeplitz operator and pseudodifferential operators are essentially the same), so in this section, we will give an outline of the calculation in this case.

Let $p(x,\xi) \in S^k(S^1) \otimes M_N$, $k > 0$, be the symbol of an $N \times N$ elliptic pseudodifferential system on the circle (we will need to assume that $[p(x,\xi), p^*(x,\xi)] = 0$ for all $x \in S^1$, $\xi \in \mathbb{R}$), and let $P_\hbar = p(x, iD/\hbar)$ be its quantization. By the McKean-Singer formula, the index of the pseudodifferential operator $P_\hbar$ (which is of course independent of $\hbar > 0$, by homotopy invariance) is equal to the trace

$$\operatorname{ind}(P_\hbar) = \operatorname{Tr}\left(e^{-P_\hbar{}^* P_\hbar} - e^{-P_\hbar P_\hbar{}^*}\right).$$

We can calculate this trace in the limit in which $\hbar \to 0$, by calculating the symbol of the operator $e^{-P_\hbar{}^* P_\hbar} - e^{-P_\hbar P_\hbar{}^*}$. The details are a bit different from the calculation of the index of a Dirac operator in [5], since there, we needed the leading order of the symbol of the heat kernel to calculate the index, whereas here, we will need the subleading order, that is, the coefficient of $\hbar$ in the asymptotic expansion in powers of $\hbar$.

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