INEQUALITIES FOR MEASURES OF SUM SETS

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1. INTRODUCTION

Suppose that E, F are Borel subsets of the middle thirds Cantor set with positive Cantor measure μ, then their sum

$$E + F = \{x + y : x \in E, y \in F\}$$

has positive Lebesgue measure. In fact Bill Moran and I showed in [4] that

$$\lambda(E+F) \geq \mu(E)^{\alpha}\mu(F)^{\alpha}, \quad \alpha = \log 3/\log 4 \tag{1}$$

(here the sets lie on the circle $\mathbb{T}$ and λ is Haar measure).

The value of α in (1) is best possible and it is precise metric results of this type that will concern us here. (Weaker conclusions of more general applicability are discussed in [3]). Attention will be restricted to sums of two sets - although (1) has recently been extended to the case where μ is replaced by the Bernoulli convolution with constant ratio of dissection m + 1. For the latter we must use m summands and α becomes $\log(m+1)/m \log 2$. See [5].

D.M. Oberlin, [10], recently gave a related result,

$$\lambda(E+F) \geq \lambda(E)^{\beta}\mu(F), \quad \beta = 1 - (\log 2/\log 3), \tag{2}$$

and I have now established the following: for $s,t \geq 1$,

$$\lambda(E+F) \geq \mu(E)^{1/s}\mu(F)^{1/t}, \quad s^{-1}+t^{-1} = \log 3/\log 2,\ 3(s+t) \leq 8; \tag{1$'$}$$

$$\lambda(E+F) \geq \lambda(E)^{1/s}\mu(F)^{1/t}, \quad s^{-1}+(\log 2/\log 3)t^{-1} = 1; \tag{2$'$}$$

$$\lambda(E+F) \geq \nu(E)^{1/s}\pi(F)^{1/t}, \quad s^{-1}+(\log 2/\log 3)t^{-1} = \log 4/\log 3; \tag{3$'$}$$