ORBIT EQUIVALENCE, LIE GROUPS, AND FOLIATIONS

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In this paper we shall survey some results on orbit equivalence of measure preserving actions of Lie groups and indicate some recent developments concerning the relationship of this classical area in ergodic theory with certain standard problems in the geometry of foliations of compact manifolds.

Suppose G is a separable locally compact group, and that G acts in a measure class preserving way on a (standard) measure space S. Then the orbits of the action define a measurable equivalence relation R(S,G) on S. Two such actions (of possibly different groups) are called orbit equivalent if the equivalence relations are isomorphic (modulo null sets.) I.e., for actions of G on S and G' on S', there is a measure space isomorphism $h:S \to S'$ such that for (almost) all $s \in S$, $h(sG)=h(s)G'$. It is also convenient to consider the more general notion of stable orbit equivalence. The actions of G on S and G' on S' are called stably orbit equivalent if the action of $G \times K$ on $S \times K$ and the action of $G' \times K$ on $S' \times K$ are orbit equivalent, where K is the circle group acting on itself by translation. (In a similar way we can speak of two equivalence relations being stably isomorphic, so that stable orbit equivalence of actions is simply stable isomorphism of the corresponding equivalence relations.) If G and G' are continuous (i.e., non-discrete) groups, and the actions are essentially locally free (i.e., almost every stabilizer is discrete), then by results of [FHM], the actions are stably orbit equivalent if and only if they are orbit equivalent. Moreover, if G and G' are connected continuous groups and the actions have fixed point sets of measure 0, then the actions are orbit equivalent if and only if they are stably orbit equivalent. We say that G is (stably) weakly equivalent to G' if they